Viscous Flow in Multiparticle Systems: Slow Motion of Fluids Relative to Beds of Spherical Particles

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A mathematical treatment is developed on the basis that two concentric spheres can serve as the model for a random assemblage of spheres moving relative to a fluid. The inner sphere comprises one of the particles in the assemblage and the outer sphere consists of a fluid envelope with a "free surface." The appropriate boundary conditions resulting from these assumptions enable a closed solution to be obtained satisfying the Stokes-Navier equations omitting inertia terms. This solution enables rate of sedimentation or alternatively pressure drop to be predicted as a function of fractional void volume.

Comparison of the theory is made with other relationships and data reported in the literature. Of special interest is its close agreement with the well known Carman-Kozeny equation which has been widely used to correlate data on packed beds as well as sedimenting and fluidized systems of particles. This is remarkable in view of the fact that the force on each particle in a packed bed can be up to several hundred times that exerted on a single particle in an undisturbed medium.

The motion of particles relative to a fluid is often of interest in the two limiting cases where either the particles move and there is no average motion of the fluid or alternatively the particles remain more or less stationary and fluid passes around them. In the problems considered here it is presumed that the effect of walls of the containing vessel is sufficiently small so that these two cases or intermediate ones, where both particles and fluid move, are mathematically equivalent. In the case of stationary fluid one is interested in predicting the effect of concentration of particles on their rate of steady sedimentation under the influence of gravity. In the case of a fluidized bed both the teeter velocity and pressure drop are of interest. In fixed beds the particles are supported by interparticle contact so that only resistance to fluid flow is involved.

This investigation is one of a series involving the fundamentals of the hydrodynamic relations of particulate systems at low Reynolds numbers (*14, 15*). The model assumed here is confined to particles which may be represented by uniform size, smooth, solid spheres. The Stokes-Navier equations omitting inertia terms are taken to describe the fluid motion. A random assemblage is considered to consist of a number of cells, each of which contains a particle surrounded by a fluid envelope which contains the same amount of fluid as the relative volume of fluid to particle volume in the entire assemblage. These envelopes will be distorted but it is assumed that a typical cell can be assumed to be spherical. Furthermore, the outside surface of each cell is assumed to be frictionless, the sedimentation case being used as the model. Thus, the entire disturbance due to each particle is confined to the cell of fluid with which it is associated. It is then possible to obtain a closed solution which describes the concentration dependence of rate of sedimentation.

It is of interest that essentially the same type of development gives good results for predicting the concentration dependence of the coefficient of viscosity of suspensions of particles (*13*).

A somewhat similar model for study of the velocity of steady fall of spherical particles through a fluid medium was employed by Cunningham (*7*), who assumed that each particle in a cloud would be effectively limited to motion within a concentric mass of fluid. However, he assumed that the boundary of the outside fluid envelope was solid; this corresponded in some way to the surfaces of the other spheres present in the cloud. This model presents the difficulty that the size of the spherical envelope must be fixed by some additional empirical consideration.

A model closer to the one employed here was used by Uchida (*26*), who assumed that the particles in a cloud would be arranged in a cubic lattice, so that each sphere would be restricted to motion in a cubic envelope of fluid in the case of sedimentation. The boundary value problem of satisfying appropriate conditions simultaneously on spherical and cubic surfaces proved intractable and consequently Uchida simplified it. The resulting solution, however, did not agree very well with experimental data.

Since the pioneer experimental investigation of Darcy (*8*) over a hundred years ago on resistance to flow of spring water through porous sands, the simple linear relationship which he obtained between rate of flow and pressure drop has presented a challenge for theoretical investigations. Numerous theoretical as well as experimental studies on the behavior of particulate systems are reviewed in detail elsewhere (*12, 15*). Some of these theories predict resistance over wide concentration ranges. However, they all involve assumptions of a semiempirical nature which are difficult to justify theoretically. In addition they usually do not give satisfactory results over the entire concentration range ($\epsilon \simeq 0.2$ to $\epsilon = 1.0$). What is still needed is a straightforward theoretical treatment which resorts to a minimum of simplifying assumptions and which satisfactorily predicts resistance over the complete range of concentration.

MATHEMATICAL DEVELOPMENT

Two concentric spheres are considered, the inner one with a radius a is solid, the outer sphere of radius b of fluid is concentric. The boundary value problem to be solved involves satisfying the creeping-motion equations and appropriate boundary conditions. It may be stated as follows:

$$\mu \nabla^2 \mathbf{v} = \text{grad}\, p \tag{1}$$

$$\text{div}\, \mathbf{v} = 0 \tag{2}$$

where $\mathbf{v}$ is the fluid velocity in the spherical shell and p the pressure at any point. The internal sphere moves in a positive direction along the x-axis with a velocity V inside a fluid sphere with a free surface and thus:

$$u = V, \quad v = w = 0 \quad \text{at} \quad r = a \tag{3}$$

$$\left.\begin{aligned} &v_r = 0, \\ &p_{r\theta} = \mu\left(\frac{\partial v_r}{r\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right) = 0 \end{aligned}\right\} \text{at } r = b \tag{4}$$

The condition of no tangential stress component on the surface of the outer sphere corresponds to vanishing of the stress tensor components $p_{r\theta}$ and $p_{r\phi}$ (*22*). The condition $v_r = 0$ corresponds to no flow across the boundary of the fluid envelope. Because of symmetry $v_\phi = 0$ in the entire spherical shell $a \leq r \leq b$. At $r = b$, $\partial v_r/\partial\theta = 0$ and so the vanishing of $p_{r\theta}$ corresponds to $\partial v_\theta/\partial r - v_\theta/r = 0$.

A general solution of the creeping-motion equations is given by Lamb (*18*). In slightly modified form, employing vector notation (*2*) it is as follows:

$$\mathbf{v} = \sum_{n=-\infty}^{n=\infty} \text{curl}\,(\mathbf{r}\chi_n) + \text{grad}\, \Phi_n + \frac{n+3}{2(n+1)(2n+3)} r^2 \,\text{grad}\, p_n - \mathbf{r}\frac{n p_n}{(n+1)(2n+3)} \tag{5}$$

$$p = \mu_0 \sum_{n=-\infty}^{n=\infty} p_n \tag{6}$$

The drag on a sphere in a fluid field is given by

$$\mathbf{W} = -4\pi\mu_0 \operatorname{grad} (r^3 p_{-2}) \qquad (7)$$

Solution is also possible with the Stokes stream function, but for consistency with the companion treatment of viscosity (*13*), the development in spherical harmonics is used here.

An appropriate form for satisfying the present boundary value problem exactly results from setting $\chi_n = 0$ and retaining solid spherical harmonics, p_n and Φ_n of orders -2 and $+1$.

$$\mathbf{v} = \operatorname{grad} \Phi_1 + \operatorname{grad} \Phi_{-2} + \tfrac{1}{5} r^2 \operatorname{grad} p_1 - \tfrac{1}{10} \mathbf{r} p_1 + \tfrac{1}{2} r^2 \operatorname{grad} p_{-2} + 2\mathbf{r} p_{-2} \qquad (8)$$

By symmetry it is found that

$$\Phi_1 = Ax$$

$$p_1 = Bx$$

$$\Phi_{-2} = \frac{Cx}{r^3} \qquad (9)$$

$$p_{-2} = \frac{Dx}{r^3}$$

With the boundary conditions expressed in Equations (3) and (4), it is found that four independent relationships are available for determining the constants A, B, C, and D. For the present problem it is only necessary to determine D corresponding to the p_{-2} harmonic required to compute the drag from Equation (7). It is as follows:

$$D = \frac{aV(3 + 2\gamma^5)}{2 - 3\gamma + 3\gamma^5 - 2\gamma^6} \qquad (10)$$

where $\gamma = a/b$. For a sphere in a fluid cell moving in the x-positive direction the drag $W_x = -4\pi\mu_0 D$ from Equations (7) and (9). For the model under consideration, the drag divided by the cell volume $(4/3)\pi b^3$ will equal $-\Delta p/L$, the pressure drop per unit length of bed. Use of this relationship gives

$$V = \left[\left(\frac{3 - \frac{9}{2}\gamma + \frac{9}{2}\gamma^5 - 3\gamma^6}{3 + 2\gamma^5}\right) \cdot \left(\frac{2a^2}{9\gamma^3\mu_0}\right)\right] \frac{\Delta p}{L} \qquad (11)$$

where the expression in brackets is the permeability coefficient in Darcy's Law.

For convenience in comparison the analogous expressions based on a dilute medium where Stokes Law applies may be employed. A sphere in an infinite medium will experience a drag,

$$W_x = -6\pi\mu_0 a V_0 \qquad (12)$$

Where wall effects can be neglected the pressure drop relationship obtained by simply adding the resistances due to individual spheres on the assumption of no interaction, Δp_0 will be given by

$$V_0 = \frac{2}{9}\left(\frac{a^2}{\gamma^3\mu_0}\right)\frac{\Delta p_0}{L} \qquad (13)$$

Equations (11) and (13) may be combined to give

$$\frac{V}{V_0} = \left(\frac{3 - \frac{9}{2}\gamma + \frac{9}{2}\gamma^5 - 3\gamma^6}{3 + 2\gamma^5}\right)\frac{\Delta p}{\Delta p_0} \qquad (14)$$

For sedimentation problems $\Delta p = \Delta p_0$ and the velocity V is the actual velocity of fall of an assemblage of spheres in a fluid which does not move. For the calculation of resistance to flow of porous media $V = V_0$ and in this case V is the superficial velocity of the fluid passing through an assemblage.

In practical problems it is often convenient to express V/V_0 dependence in terms of the solids concentration $c = \gamma^3$ or void volume fraction $\epsilon = 1 - c$. Some typical values of $(V/V_0)_{\Delta p = \Delta p_0}$ are given in Table 1.

In the rest of this paper, values of $(V/V_0)_{\Delta p = \Delta p_0}$ will be listed without the subscript to avoid additional symbolism. This usage is consistent with other treatments appearing in the literature. It is, of course, evident that $(V/V_0)_{\Delta p = \Delta p_0}$ will be equal to $(\Delta p_0/\Delta p)_{V = V_0}$.

Table 1.

Theoretical Relative Velocity Function

Sphere radius ratio, $a/b = \gamma$	Solids conc., vol. fraction $\gamma^3 = c = 1 - \epsilon$	Relative velocity $(V/V_0)_{\Delta p = \Delta p_0}$
0.1	0.001	0.8500
0.2154	0.01	0.6773
0.3684	0.05	0.4526
0.4642	0.10	0.3215
0.5848	0.20	0.1773
0.6694	0.30	0.09867
0.7368	0.40	0.05287
0.7937	0.50	0.02638
0.8434	0.60	0.01175
0.8879	0.70	0.004340
0.9283	0.80	0.001132

VALIDATION OF FREE SURFACE MODEL THEORY

Dilute Range

At the dilute end of the solids concentration range, sedimentation data are available from a study by McNown and Lin (*20*) for a range of concentration from 0.1%-vol. ($c = 0.001$) to 3%-vol. of solids. Their data were obtained at a particle Reynolds number with respect to fluid of approximately unity with the use of carefully graded particles of sand and glass spheres of about 100 μ diam. A correction factor was developed which allows for the small difference at a Reynolds number approaching zero. Thus, for example, V/V_0 was observed to be 0.78 at a 1%-vol. solids concentration, with a corrected value at zero Reynolds number of $V/V_0 = 0.74$. Table 2 gives a comparison of results from this study with values predicted by Equation (14).

Table 2.

Sedimentation at Low Concentrations

Solids conc. %-vol.	Relative McNown	velocity, V/V_0 Eq. (14)
0.1	0.87	0.85
0.5	0.80	0.74
1.0	0.74	0.68
2.0	0.70	0.60

Data shown in Table 2 are also confirmed by a theoretical development presented by McNown and Lin which is in agreement with that of Burgers (*4*) and Smoluchowski (*24*) for cubical arrangements of the particles. For low Reynolds numbers the McNown and Lin theory gives $V/V_0 = 1/(1 + 1.6c^{1/3})$, which is not far from the effect predicted by Equation (14), $V/V_0 = 1 - 1.5c^{1/3}$ when $c < 0.001$. Uchida's theory (*26*) also results in a similar relationship at low concentrations, namely $V/V_0 = 1/(1 + 2.1c^{1/3})$. These relationships* all indicate a rapid decrease in relative velocity when only a few tenths of a percent of solids are present.

There are several other theoretical relationships (*16*) including one by Burgers (*4*) which involves the first power of ϕ instead of its cube root. For this case Burgers assumes that all particles will be free to occupy a random position around a given reference particle and obtains $V/V_0 = 1/(1 + 6.88c)$. It is difficult to see why random arrangement should result in entirely different depen-

*Another recent treatment [Mitutosi Kawaguti, *J. Phys. Soc. of Japan*, 13, 209 (1958)] applies viscous flow past a sphere in a frictionless circular pipe to sedimentation. In dilute systems it gives

$$V/V_0{}^{n/-1.6c^{1/3}}$$

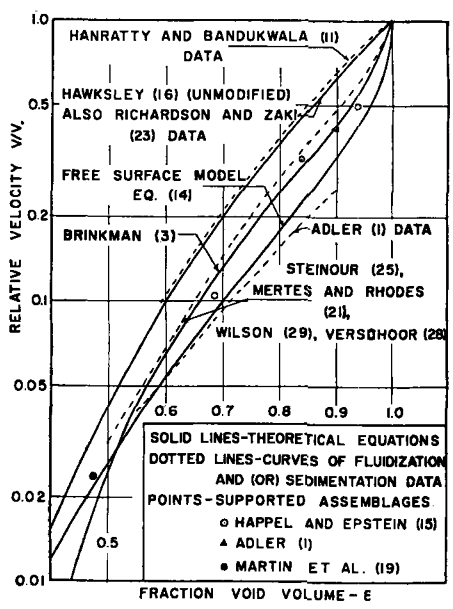

Fig. 1. Comparison of theoretical relationships with data in intermediate concentration range.

dence of sedimentation rate in concentration, but further work will be needed to explain the discrepancy.

Another complication encountered in very dilute assemblages is the effect of the walls of the containing vessel on sedimentation velocity. This has been discussed by Happel and Brenner (*14*) but further work is necessary to define the relative importance of particle-to-particle and particle-to-wall interactions. It would also be desirable to have more extensive experimental data in the dilute region to test the various theories advanced for the effect of concentration, both with and without wall effects.

Intermediate Concentration Range

In the range of concentrations from $\epsilon \simeq 0.05$ to $\epsilon \simeq 0.4$ most of the data reported are for fluidized or sedimenting systems. Figure 1 gives a representative collection of such data in this range along with several theoretical relationships. These data are for liquids and uniform spheres at particle Reynolds numbers in the slow viscous flow region ($N_{Re} <$ 0.3). There is considerable lack of agreement.

The highest values of relative velocity are exhibited by the experiments of Hanratty and Bandukwala (*11*) in which glass beads are fluidized and sedimented in viscous liquids. The similar data of Richardson and Zaki (*23*) agree closely. Both sets of data are in agreement with the theoretical treatment of Hawksley (*16*) and close to the theoretical relationship developed by Richardson and Zaki (Configuration 2—not plotted). These data, however, average about 35% higher in V/V_0 than most of the other fluidization and sedimentation data reported.

The latter include the data of Steinour (*25*) on sedimentation, that of Mertes and Rhodes (*21*) on fluidization and sedimentation, and the fluidization experiments of Verschoor (*28*) on closely graded glass spheres in toluene. Verschoor conducted other experiments on the fluidization of sand particles in water which give average results closer to the theoretical curve of Brinkman which is 15% below the average of this set of data.

A limited amount of data has been recently obtained by Adler (*1*) on fluidization of glass spheres in glycerine in an apparatus much the same as employed by Happel and Epstein (*15*). The purpose of this work was to study the reasons for the variation in results noted above. Adler employed a conical-type screen at the bottom of the bed so that bed inlet flow was greater along the walls than the center of the column. He also employed bed height to diameter ratios smaller ($L/D = 1/1$ to $6/1$) than have usually been used in previous studies. In this way it was possible to minimize circulation and particle segregation effects. The data obtained show V/V_0 values considerably lower than reported in other investigations but are not far from agreement with the "free surface" model developed in this paper.

Other studies have been conducted on supported assemblages of spherical particles in order to eliminate the effect of possible changes in relative particle position during fluidization or sedimentation. These data, all on cubic assemblages, along with the packed-bed data of Martin (*19*) corresponding to cubic arrangement are shown as individual points on Figure 1. They are not far from agreement with the Brinkman and "free surface" model relationships. Of course, a cubic arrangement may give different results from the arrangement in an actual bed even if segregation does not occur. The relative velocity on these assemblages was evaluated from pressure drop data on the assumption that wall effect would not be important and in the dilute region this might be a factor. Wilson (*29*) was able to prepare more or less random assemblages of spheres without the need for support up to values of $\epsilon = 0.84$ by using dilute gelatin to promote adhesion between particles. He obtained permeabilities in good agreement with the Brinkman's equation.

It appears that a unique relationship between relative velocity and fraction void volume does not exist in this concentration range. The "free surface" model does give a lower bound and perhaps corresponds to the condition representing uniform particle distribution since any circulation or agglomeration effects will result in increased values for relative velocity. It is plausible to suppose that agglomeration effects will be of greatest importance in the intermediate concentration region. In very dilute systems particles will not be close enough to attract each other, whereas in concentrated systems mutual interference will result in a more or less uniform development of structure.

High Concentration Range

In the more concentrated range the most striking confirmation of the theory developed here is its close correspondence with the Carman-Kozeny (*5, 17*) or Fair-Hatch (*9*) equation, widely employed for the correlation of flow data for particulate systems. This equation is based on a semi-theoretical approach which assumes that pore space is equivalent to a bundle of parallel capillaries with a common hydraulic radius. In the terminology used here it takes the form

$$\frac{V}{V_0} = \frac{1}{2k} \frac{\epsilon^3}{1 - \epsilon} \left(\frac{\Delta p}{\Delta p_0} \right) \qquad (15)$$

where the constant k is to be evaluated experimentally. In the case of consolidated media where the solid forms a continuous structure, this equation does not apply well as might be expected. Also as $\epsilon \rightarrow 1$, it requires indefinitely high values of k. In a book devoted to the flow of gases through porous media Carman (*6*) discusses many aspects of the applicability of this equation which is supported by a considerable mass of experimental data.

For packed beds of uniform spheres with a range of $\epsilon = 0.26$ to $\epsilon = 0.48$, he concludes that the best correlation corresponds to $k = 4.8$, with a probable range of variation due to experimental uncertainty of 4.5 to 5.1. A large volume of data on beds consisting of a number of other shapes of particles, some of rather extreme type, indicate that $k \simeq 5.0$ independent of shape and porosity from $\epsilon = 0.26$ to $\epsilon = 0.8$. This indicates that as long as particles are maintained in a relatively fixed position with respect to each other departure from tne smooth, spherical shape can be substantial without changing relative velocity V/V_0 at a given voidage.

The Carman-Kozeny relationship with $k = 5.0$ is plotted on Figure 2 along with Equation (14) for comparison The agreement is remarkable, the average deviation of Equation (14) in the range $\epsilon = 0.2$ to $\epsilon = 0.7$ being 9%. The maximum deviation is 13% at $\epsilon = 0.70$ and the curves cross at $\epsilon = 0.58$. With $k = 4.8$, the average deviation is 8% over the same range. Maximum deviation is 17% at the upper end $\epsilon = 0.70$ and the curves cross at about $\epsilon = 0.52$. Much better agreement is, of course, possible by omitting the $\epsilon = 0.70$ point but even it is within experimental error.

Of particular interest, in connection with the study of relationships between flow in packed beds and the closely related phenomena of sedimentation and fluidization is the loose-packed condition (*12*) corresponding to $\epsilon \simeq 0.47$. This point corresponds to the void volume for incipient fluidization and also to the void volume for moving beds. The value of V/V_0 from the Carman-Kozeny equation with $k = 4.8$ for this point (actually setting $\epsilon = 0.476$ corresponding to a simple cubic packing because loose-packed voidage is not known to three significant figures) corresponds to 0.0216 as compared with 0.0221 from Equation (14), a difference of less than 3%.

A study on flow through beds of regularly stacked spheres by Martin, McCabe and Monrad (*19*) indicates the effect of particle orientation on resistance. Four porosities were considered, namely cubic packing, $\epsilon = 0.476$; orthorhombic, $\epsilon = 0.395$; tetragonal spheroidal, $\epsilon = 0.302$; and rhombohedral (close packing), $\epsilon = 0.260$. The loosest packing, cubic, corresponds closely in void volume to that of a random loose-packed bed, just discussed. The relative velocity $V/V_0 = 0.0235$ is only 6.5% higher than the value from Equation (14), lending weight to the hypothesis that a cubic assemblage can serve as a model for a random bed in viscous flow. The two intermediate porosities show some disagreement but the extreme case of rhombohedral packing gives an experimental value of $V/V_0 = 0.00251$ as compared with a value from Equation (14) of 0.00267. This corresponds to the lowest porosity possible with a uniform-sized spherical particle bed. The resistance to flow is just four

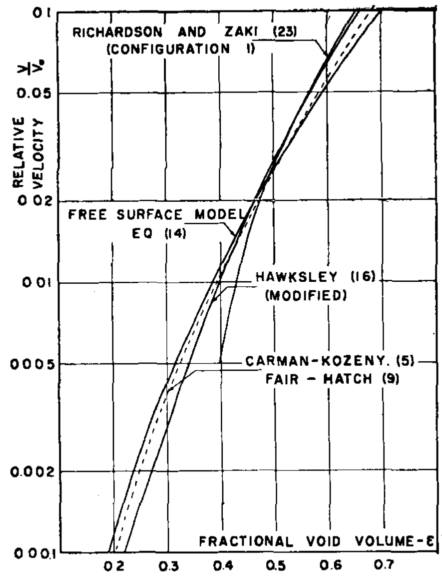

Fig. 2. Comparison of theoretical relationships with Carman-Kozeny equation in high concentration ranges.

hundred times as great as that for a single sphere but the difference between theoretical and predicted result is less than 10%.

As can be seen from Figure 2, the other theoretical relationships do not give as good agreement with the Carman-Kozeny equation as does the free surface model, especially at values of ϵ less than 0.5.

DISCUSSION OF THEORETICAL RELATIONSHIPS

The theoretical studies of most interest are those plotted on Figures 1 and 2. At about $\epsilon = 0.5$ they are in close agreement with each other, and the data even though based on quite different basic assumptions, except that all do involve the creeping-motion equations and uniform spherical particles. It is interesting to note that no smooth transition seems to be able to account simultaneously for the high V/V_0 values shown for some fluidization and sedimentation data and the relatively low V/V_0 values in the packed-bed region.

One of the most widely quoted theories aimed at extending empirical relationships to the more dilute region is that of Brinkman (*3*). His relationship for relative velocity corresponds to

$$\frac{V}{V_0} = 1 + \frac{3}{4}(1 - \epsilon) \cdot \left(1 - \sqrt{\frac{8}{(1 - \epsilon)} - 3}\right) \qquad (16)$$

The model underlying his treatment is that of a spherical particle embedded in a porous mass. The flow through this porous mass is described by a modification of Darcy's (*8*) equation and since the latter is empirical the result cannot be considered a rigorous solution to the problem. Furthermore, when $\epsilon = \frac{1}{3}$ the Brinkman equation gives a value of $V/V_0 = 0$, corresponding to zero permeability.

Richardson and Zaki (*23*) developed a cell-type model for sedimenting spherical particles in which the particles were assumed to be spaced in a hexagonal-type pattern in the horizontal direction. Particles were assumed to be settling in such a way that they were all lined up vertically above each other, but the distance between adjacent horizontal layers was an additional variable parameter and two cases were worked out. The specification of this distance which gave the best agreement with Richardson and Zaki's own data in the intermediate concentration range is designated by them as Configuration 2. It assumes that the spheres in adjacent horizontal layers *actually touch.* Configuration 1, which is plotted in Figure 2, assumes that the spheres are the same distance apart vertically as their distance from each other in the horizontal hexagonal pattern. As would be expected this uniform spatial distribution gives lower values of V/V_0 for corresponding values of fractional void volume. In fact the result is in close agreement with Brinkman's correlation in the intermediate concentration range and like Brinkman's shows considerable divergence from the packed-bed data at values of ϵ below 0.45. Due to failure to solve the specified boundary value problem exactly the relative velocity becomes infinite at infinite dilution ($\epsilon = 1.0$), instead of the correct value of $V/V_0 = 1.0$.

The treatment of Hawksley (*16*), although giving good agreement with data, is the least rigorous of those considered here in that no clearly defined boundary value problem is postulated. It is based on the assumption that the resistance experienced by a particle in a suspension can be approximated by the effective local resistance to shearing that would arise if there were a small relative displacement between the given particle and the rest of the suspension. This assumption is a tenable one, but the problem of evaluation of the motion of the particle with respect to the rest of the suspension is still not simple. Hawksley assumes that the drag on the given representative sphere can be evaluated using Stokes Law with a modified viscosity instead of that of the pure fluid. This modified viscosity is obtained from the treatment of Vand (*27*) for the viscosity of a suspension. The Vand treatment takes into account both hydrodynamic interaction between particles and the effect of collisions between them in the shearing field. Hawksley takes only the part of the Vand treatment which considers hydrodynamic interaction which gives for the effective local viscosity μ_c in terms of pure fluid viscosity μ_0 the following:

$$\frac{\mu_c}{\mu_0} = \exp\left[\frac{2.5(1 - \epsilon)}{1 - \frac{39}{64}(1 - \epsilon)}\right] \qquad (17)$$

This effective local viscosity will not be as great as the actual viscosity as determined in a viscometer, and Vand gives a further modification accounting for particle collisions which show more marked concentration dependence. Neither of Vand's equations has been supported unequivocally by experimental data and so the choice of effective local velocity and of Equation (17) is to some extent arbitrary. The application of Stokes Law also involves using an average density for the surrounding suspension as well as an effective local viscosity. Strictly speaking, of course, Stokes Law will not apply unless the medium surrounding the chosen sphere is a continuum which in effect means that the representative sphere would be required to be much larger than those surrounding it instead of being equal in size.

With these assumptions relative velocity can be expressed as follows:

$$\frac{V}{V_0} = \frac{\epsilon^2}{\exp\left[\frac{2.5(1 - \epsilon)}{1 - \frac{39}{64}(1 - \epsilon)}\right]} = \epsilon^2\left(\frac{\mu_0}{\mu_c}\right) \qquad (18)$$

This expression agrees well with the set of data showing high V/V_0 values in the intermediate concentration range. It does not, however, give good agreement with packed-bed data.

Hawksley finds that in order for his original equation to agree with packed-bed data in the concentration range $\epsilon = 0.2$ to $\epsilon = 0.6$ it is necessary to apply an orientation factor. This factor is taken as $\delta = (\sin^2 \omega)_{avg} = \frac{2}{3}$, where ω is the angle between the normal to an element of surface of the particle and the direction of average motion relative to the particle. The factor δ is not independent of concentration. It is stated that as concentration decreases, the fluid streamlines will become more nearly parallel to the direction of bulk flow and δ will tend to unity. This factor certainly is not clearly established but the need for it does demonstrate the discontinuity between the data exhibiting high V/V_0 values in the intermediate concentration range and that for packed beds.

It is of interest to examine further theoretical implications in two of Hawksley's postulates in the light of alternative justification because of the good agreement of his treatment with data. The assumption that Equation (18) applies can be conveniently examined by means of the uniform viscosity treatment developed on the basis of the same mathematical model as used in the present paper. Table 3 presents an appropriate comparison with the use of values for μ/μ_0 from this model (13), which also show good agreement with data at high concentrations.

TABLE 3.
RELATIONSHIP BETWEEN RELATIVE VELOCITY (V/V_0) AND RELATIVE VISCOSITY (μ/μ_0)

Solids conc., c	Rel. Vis. (*13*) μ/μ_0	Predicted rel. velocity A V/V_0 (Eq. 18) $(1-c)^2/(\mu/\mu_0)$	B V/V_0 (Table 1)	Ratio (A/B)
0.05	1.281	0.702	0.453	1.55
0.10	1.605	0.502	0.3215	1.55
0.20	2.486	0.257	0.1773	1.45
0.30	3.942	0.124	0.09867	1.25
0.40	6.621	0.0544	0.05287	1.03
0.50	12.20	0.0205	0.02638	0.79

It is seen that there is indeed an approximate correspondence, justifying the employment of Equation (18), and that furthermore it gives predicted relative velocities in the dilute region which are higher than those determined by a consistent mathematical model, although in the right direction to match experimental data in the high V/V_0 range obtained for some fluidized systems.

Another justification for the orientation factor δ, which appears useful to correlate data on some fluidized and sedimenting systems, might be that agglomeration occurs in such systems. A doublet consisting of two spheres following each other and touching will acquire a relative velocity $V/V_0 = 1.55$; that is, it will fall 55% faster than the spheres falling separately (*10*). Thus in a sufficiently dilute assemblage consisting of such doublets the value of $V/V_0 = 1.55$ would be approached at zero concentration instead of $V/V_0 = 1.00$ corresponding to Stokes Law. This corresponds to an orientation factor $\delta = 0.645$ instead of Hawksley's $\delta = 0.667$ based on streamline direction considerations.

SUMMARY

The free surface model adopted here for mathematical development results in a relationship, Equation (14), which applies to a wide variety of flow data through porous media. In the highly concentrated range usually encountered in packed beds ($\epsilon \simeq 0.2$ to 0.6) it gives results within experimental accuracy of those reported in the literature. In the intermediate concentration range, usually applying in fluidized and sedimenting systems of particles ($\epsilon \simeq 0.6$ to 0.95), it gives results which are in agreement with fluidization data taken under special conditions to minimize particle-segregation effects. However these values of V/V_0 are from 25–100% below various experimental data reported in the literature, indicating that V/V_0 is not uniquely characterized by fractional void volume. Comparison with the data of McNown and Lin in the dilute range ($\epsilon > 0.98$) show Equation (14) to be low to the extent of only 2–15%, probably within experimental accuracy. Best agreement is in the extremely dilute ($\epsilon = 0.999$) region.

As regards the derivation, it is based on a simple modification of the Stokes-Navier equations of motion, postulates a clearly defined boundary value problem, and gives an exact mathematical solution to this problem. The solution does not result in impossible predictions at either the dilute or concentrated ends. It involves no arbitrary parameters to be applied to different portions of the concentration range or for conditions involving other variables than void volume fraction.

It thus presents a simple mathematical model which should serve as a basis for extension to more complicated cases. These might include studies taking into account such items as inertial effects, fluid interchange between cells, variations in particle shape (including cylindrical), variations in particle distribution such as occur in fluidized and sedimenting systems. Work is now in progress on approximations for simultaneous heat or mass transfer as well as chemical reaction effects with use of the free surface model. It may also be useful in development of empirical relationships (*30*) often employed in practical application of particulate systems.

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NOTATION

a = radius of solid sphere
b = radius of fluid sphere
c = solids concentration
A, B, C, D = constants
p = pressure
Δp = pressure drop through a suspension
Δp_0 = pressure drop through a suspension calculated on basis of Stokes Law
$p_{r\theta}$ = stress component in θ direction across plane perpendicular to r
$p_{r\phi}$ = stress component in ϕ direction across plane perpendicular to r
p_n = solid spherical harmonic of order n
r = radial distance from origin
u = velocity in x direction
$\mathbf{v}$ = velocity vector
v = velocity in y direction
v_r = velocity in radial direction
v_θ = velocity in θ direction
V = superficial velocity or velocity of sedimentation in x direction
V_0 = sedimentation velocity of sphere with radius a in infinite medium
w = velocity in z direction
$\mathbf{W}$ = drag vector
W_x = drag in x direction
x = distance from origin in x direction

Greek Letters

γ = ratio of a/b
δ = orientation factor
ϵ = fractional void volume
μ = viscosity of a suspension
μ_e = effective local viscosity of a suspension
μ_0 = viscosity of pure fluid
χ_n = solid spherical harmonic of order n
Φ_n = solid spherical harmonic of order n
ω = angle between normal to an element of surface and direction of particle motion

Note: Spherical coordinates r, θ and ϕ employed here are related to Cartesian coordinates by the relations $x = r \cos\theta$; $y = r \cos\phi \sin\theta$; $z = r \sin\phi \sin\theta$.

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